RG Flow of Magnetic Brane Correlators

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Abstract

The magnetic brane solution to five-dimensional Einstein-Maxwell-Chern-Simons theory provides a holographic description of the RG flow from four-dimensional Yang-Mills theory in the presence of a constant magnetic field to a two-dimensional low energy CFT. We compute two-point correlators involving the U(1) current and the stress tensor, and use their leading IR behavior to confirm the existence of a single chiral current algebra, and of left- and right-moving Virasoro algebras in the low energy CFT. The common central charge of the Virasoro algebras is found to match the Brown-Henneaux formula, while the level of the current algebra is related to the Chern-Simons coupling. The coordinate reparametrizations produced by the Virasoro algebras on the AdS_3 near-horizon geometry arise from physical non-pure gauge modes in the asymptotic AdS_5 region, thereby providing a concrete example for the emergence of IR symmetries. Finally, we interpret the infinite series of sub-leading IR contributions to the correlators in terms of certain double-trace interactions generated by the RG flow in the low energy CFT.

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1. Introduction

A powerful feature of the AdS/CFT correspondence is that it provides a geometrical description of RG flow, via the emergence of an extra holographic radial direction in the bulk spacetime. An RG flow between UV and IR CFTs is mapped to a bulk geometry that interpolates between one asymptotic AdS geometry at large radial coordinate r, and another at small r.

The UV and IR CFTs may differ in their spacetime dimensionality. The IR dimensionality is reduced compared to that in the UV if the low energy excitations are confined to propagate in only some of the directions of the original space. A simple example of this phenomenon occurs for charged particles in a strong magnetic field. Semi-classically, the particles undergo circular motion at the cyclotron frequency around the magnetic flux lines, and the low energy excitations correspond to the drift velocity parallel to the field. The same picture emerges from Landau level quantization, with the result that the effective dimensionality of the system at low energies is reduced by two, compared to that at high energies. A D=3+1 CFT in the presence of a magnetic field thus flows to a D=1+1 CFT at low energies[4]

A bulk geometry dual to such an RG flow was found in [1].[5] This geometry was obtained as a solution to D=4+1 Einstein-Maxwell-Chern-Simons theory, and interpolates between AdS_5 at large r, and $\mathrm{AdS}_3 \times R^2$ at small r. The spacetime is supported by a constant field strength filling two spatial dimensions. As discussed in [1], these solutions include holographic duals to D=3+1 super-Yang-Mills theories, including the maximally supersymmetric $\mathcal{N} = 4$ theory, in the presence of an external magnetic field coupling to their R-current. Finite density generalizations of these solutions were obtained and studied, both numerically and analytically, in [6], [7], [8] (see also [9]), and a rich structure was discovered, including the existence of a quantum critical point with non-trivial critical exponents.

Given an RG flow, it is interesting to explore how the IR CFT emerges from the structure of correlation functions computed in the full theory. One may expect that, in the low energy limit, the correlation functions should be describable in terms of an IR CFT, governed by the dual near-horizon geometry. It has been pointed out recently in [10], [11], and [12] however, that this description of IR behavior may be incomplete when gapless degrees of freedom (or Goldstone bosons) exist which are supported in the bulk, for finite r. These issues can be studied quite concretely and explicitly using holographic RG flow solutions.

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[4] Strictly speaking, the two spatial dimensions should be compact in order for the IR CFT to have a finite total central charge, rather than a central charge per unit area.

[5] An earlier related example is studied in [2], and some generalizations appear in [3]. We also note that correlators in RG flows to spacetimes with an AdS_2 factor have been studied recently in connection with models of non-Fermi liquids; see, for example, [4], [5].

In this paper we compute low energy two-point correlation functions for the current and the stress tensor, dual to perturbations of the gauge field and metric of the bulk gravity solution found in [1]. We work at zero temperature. The bulk solutions, obtained in [1] involve some numerical input. But as we will see, enough is known about the asymptotics of the solutions to obtain analytic results for the low energy correlation functions.

On general grounds, we expect that the stress tensor two-point function should be consistent with the existence of a D=1+1 CFT with a central charge given by the Brown-Henneaux formula [13]. In particular, we should be able to infer the existence of a pair of Virasoro algebras generated by the modes of the stress tensor. In the standard Brown-Henneaux analysis for AdS_3, the Virasoro generators implement coordinate transformations that act nontrivially on the AdS_3 boundary. On the other hand, in our analysis, the stress tensor is defined at the AdS_5 boundary. This leads to a small puzzle, in that the standard asymptotic boundary conditions for AdS_5 are only compatible with a finite dimensional group of asymptotic coordinate transformations. So how can the infinite dimensional algebra of Brown-Henneaux coordinate transformations manifest itself in terms of the AdS_5 stress tensor? We answer this question by explicit computation. The key point turns out to be that the Brown-Henneaux coordinate transformations defined in the near horizon AdS_3 region turn into physical (non pure-gauge) modes when extended to the AdS_5 region. The relationship between the AdS_3 and AdS_5 stress tensors is then found to be precisely what is needed in order for a Virasoro algebra to emerge at the AdS_5 boundary. Understanding gained from this example should be useful more generally in understanding how emergent symmetries arise in the IR.

Another set of issues arises when we consider correlation functions of the boundary current $\mathcal{J}_\pm$. The bulk gauge field is governed by Maxwell and Chern-Simon terms, with the latter leading to a chiral anomaly for the dual boundary current. The leading long distance behavior of the current correlator is constrained by the need to saturate the chiral anomaly. To leading order in the IR, the two-point functions $\langle \mathcal{J}_- \mathcal{J}_\pm \rangle$ are subdominant so that $\langle \mathcal{J}_+ \mathcal{J}_+ \rangle$ by itself must saturate the chiral anomaly. This forces $\langle \mathcal{J}_+ \mathcal{J}_+ \rangle$ to have a $1/(x^+)^2$ falloff[6] with a specified coefficient, the sign of which is mandated by unitarity. (Reversing the sign of the Chern-Simons coupling will reverse the chiralities of the currents, but will maintain the sign of this coefficient, in keeping with unitarity.) At intermediate momenta, large compared with the AdS_3 scale but small compared to the AdS_5 scale, the situations is reversed: $\langle \mathcal{J}_+ \mathcal{J}_\pm \rangle$ are now subdominant, and it is $\langle \mathcal{J}_- \mathcal{J}_- \rangle$ that saturates the anomaly. The sign of the coefficient in this correlator is now opposite to the one required by "naive unitarity" in this intermediate regime. Unitarity is maintained, however, in the full magnetic brane solution with AdS_5 asymptotics. We show that all these properties are indeed borne out in our holographic calculations.

In the IR behavior of the current correlator $\langle \mathcal{J}_+ \mathcal{J}_+ \rangle$, we also find an infinite series of

[6] Throughout, $x^\pm$ refer to light-cone coordinates along the D=1+1 boundary directions, while $\pm$ refer to the corresponding Einstein indices.

sub-leading terms, whose analogues also appear in the correlators $\langle \mathcal{J}_- \mathcal{J}_\pm \rangle$. We show how these sub-leading terms can be understood in terms of double trace interactions generated in the RG flow towards the IR fixed point CFT. Such effects have been discussed recently in [11] and [12]. The double trace interaction involves a current bilinear, and as usual [14] its coupling ζ is related to the form of mixed Neumann-Dirichlet boundary conditions for the gauge field at the AdS$_3$ boundary. In fact, as far as the current correlators are concerned, all the information about how the AdS$_3$ geometry is embedded in the magnetic brane solution with AdS$_5$ asymptotics is contained in the value of the parameter ζ, whose sole dependence is on the Chern-Simons coupling.

Our computations are conceptually instructive in that they illustrate how to derive new AdS/CFT dictionaries from old ones. In particular, the holographic dictionary for a gauge field in AdS$_3$ with both Maxwell and Chern-Simons terms is subtle (and has not been written down explicitly as far as we know; see [15], [16] for the pure Chern-Simons case), due to the presence of several modes with different growth at the boundary. On the other hand, the corresponding dictionary for a gauge field in an asymptotically AdS$_5$ geometry is straightforward. By taking the low energy limit of the correlation functions, we can use the latter to derive the rules for the former.

The remainder of this paper is organized as follows. In section 2 we review the construction of the boundary stress tensor and current for our theory. In section 3 we collect needed results on the magnetic brane solution interpolating between AdS$_5$ and AdS$_3 \times R^2$. In section 4 we compute the current correlators, and discuss their interpretation in section 5. We consider the stress tensor correlators in section 6, and conclude with some comments in section 7. The analytic continuation between Minkowski and Euclidean signatures is relayed to Appendix A, while an evaluation of needed Fourier integrals may be found in Appendix B.

The computations in this paper can be extended to the case of finite charge density, and the results will appear separately [17].

2. Stress tensor and current definitions

The action of five-dimensional Einstein-Maxwell-Chern-Simons theory with a negative cosmological constant is[7]

$$S = -\frac{1}{16\pi G_5} \int_M d^5x \sqrt{g} \left(R + F^{MN} F_{MN} - \frac{12}{L^2} \right) + S_{CS} + S_{\text{bndy}} \tag{2.1}$$

where the Chern-Simons action is given by

$$S_{CS} = \frac{k}{12\pi G_5} \int_M A \wedge F \wedge F \tag{2.2}$$

[7] The action is for a metric with Minkowski signature. Our metric and curvature conventions follow those adopted by Weinberg [18].

In these conventions, the action (2.1) coincides with that of minimal $D = 5$ gauged supergravity when $k = 2/\sqrt{3}$. In this paper k will be left unspecified, although we will assume $k \geq 0$; there is no loss of generality here, since the sign of k is flipped by the field redefinition $A \to -A$. The contributions denoted by $S_{\rm bndy}$ consist of the usual boundary counterterms needed for a well defined variational principle; their explicit forms will not be needed here. We henceforth set $L = 1$.

For the supersymmetric value $k = 2/\sqrt{3}$, this action is a consistent truncation known to describe all supersymmetric compactifications of Type IIB or M-theory to AdS$_5$ (see [19], [20], and [21]). This means that solutions of (2.2) are guaranteed to be solutions of the full 10 or 11 dimensional field equations (although for non-supersymmetric solutions there is no guarantee of stability). It also implies that the solutions we find are holographically dual not just to $\mathcal{N} = 4$ super-Yang-Mills, but to the infinite class of supersymmetric field theories dual to these more general supersymmetric AdS$_5$ compactifications.

The Bianchi identity is $dF = 0$, while the field equations are given by,

$$\begin{aligned} 0 &= d \star F + kF \wedge F \\ R_{MN} &= 4g_{MN} + \frac{1}{3} F^{PQ} F_{PQ} g_{MN} - 2F_{MP} F_N{}^P \end{aligned} \tag{2.3}$$

We will be considering asymptotically AdS$_5$ solutions, in the sense that the metric and gauge field admit a Fefferman-Graham expansion [22]. Introducing a radial coordinate ρ, defined such that the AdS$_5$ boundary is located at $\rho = \infty$, the metric takes the asymptotic form

$$\begin{aligned} ds^2 &= \frac{d\rho^2}{4\rho^2} + g_{\mu\nu}(\rho, x) dx^\mu dx^\nu \\ g_{\mu\nu}(\rho, x) &= \rho g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x) + \frac{1}{\rho} g^{(4)}_{\mu\nu}(x) + \frac{\ln \rho}{\rho} g^{(\ln)}_{\mu\nu}(x) + \cdots \end{aligned} \tag{2.4}$$

and for the gauge field we have

$$\begin{aligned} A &= A_\mu(\rho, x) dx^\mu \\ A_\mu(\rho, x) &= A^{(0)}_\mu(x) + \frac{1}{\rho} A^{(2)}_\mu(x) + \cdots \end{aligned} \tag{2.5}$$

Here $x^M = (\rho, x^\mu)$, with $\mu = 0, 1, 2, 3$.

A key point in the construction of the boundary stress tensor [23][24] is that the coefficients $g^{(2)}_{\mu\nu}$ and $g^{(\ln)}_{\mu\nu}$ are fixed by the Einstein equations to be local functionals of the conformal boundary metric $g^{(0)}_{\mu\nu}$. On the other hand, for $g^{(4)}_{\mu\nu}$ the Einstein equations only fix the trace, $\mathrm{tr}({g^{(0)}}^{-1} g^{(4)})$, to be a local functional of $g^{(0)}_{\mu\nu}$.

The boundary stress tensor and current are defined in terms of the variation of the on-shell action

$$\delta S = \int d^4x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\mu\nu} \delta g^{(0)}_{\mu\nu} + J^\mu \delta A^{(0)}_\mu \right) \tag{2.6}$$

In terms of the Fefferman-Graham data the result is [23][24]

$$\begin{aligned} 4\pi G_5 T_{\mu\nu}(x) &= g^{(4)}_{\mu\nu}(x) + \text{local} \\ 2\pi G_5 J_\mu(x) &= A^{(2)}_\mu(x) + \text{local} \end{aligned} \tag{2.7}$$

where indices are lowered using the conformal boundary metric $g^{(0)}_{\mu\nu}$. The local terms denote tensors constructed locally from $g^{(0)}_{\mu\nu}$ and $A^{(0)}_\mu$. In this paper we are interested in computing two-point correlation functions of operators at non-coincident points. For this, we need to compute the stress tensor and current induced at one point by a variation of $g^{(0)}_{\mu\nu}$ and $A^{(0)}_\mu$ at a different point. The local terms in (2.7) do not contribute, and hence are not needed for computing correlators of operators at distinct points; they instead contribute to contact terms involving delta functions and derivatives of delta functions. We henceforth drop the local terms.

3. The magnetic background solution

3.1. Gravity solution

Our background solution is holographically dual to a four-dimensional CFT in a constant external magnetic field. The bulk solution takes the form

$$\begin{aligned} ds^2 &= \frac{dr^2}{L_0(r)^2} + 2L_0(r)dx^+dx^- + e^{2V_0(r)}dx^i dx^i \ , \qquad i = 1,2 \\ F &= b dx^1 \wedge dx^2 \end{aligned} \tag{3.1}$$

The value of b can be changed by rescaling x^i; a convenient choice turns out to be

$$b = \sqrt{3} \tag{3.2}$$

Inserting the *Ansatz* (3.1) into the field equations gives the following the equations,

$$\begin{aligned} & L_0'' + 2V_0'L_0' + 4(V_0'' + V_0'^2)L_0 = 0 \\ & 6L_0^2V_0'' + 8L_0^2(V_0')^2 + 4L_0L_0'V_0' - (L_0')^2 + 12e^{-4V_0} = 0 \\ & 4L_0^2(V_0')^2 + 8L_0L_0'V_0' - 24 + (L_0')^2 + 12e^{-4V_0} = 0 \end{aligned} \tag{3.3}$$

We note that there is some redundancy in this system of equations, since by using the derivative of the third equation one can show that one combination of the first two equations is obeyed identically. L_0 can be determined in terms of V_0 as

$$L_0(r)^2 = 24e^{-2V_0(r)} \int_0^r dr' \int_0^{r'} dr'' e^{2V_0(r'')} \tag{3.4}$$

The function V_0 is determined numerically; see [8].

3.2. Asymptotic behavior of the solution

As $r \to \infty$ the solution approaches AdS_5. The asymptotics are

$$\begin{aligned} L_0(r) &= 2(r - r_0) - \frac{3 \ln r}{c_V^2 r} + \mathcal{O}(r^{-1}) \\ e^{2V_0(r)} &= c_V(r - r_0) + \mathcal{O}(r^{-1}) \end{aligned} \tag{3.5}$$

If we scale x^i such that (3.2) is obeyed, and shift r such that $L(0) = 0$, then the constants c_V and r_0 are found numerically to be $c_V \approx 2.797$, $r_0 \approx 0.53$. The AdS_5 radius is $L_5 = L = 1$.

The coordinate r differs from the coordinate ρ appearing in the Fefferman-Graham expansion. At large r they are related by

$$\rho = 4(r - r_0) + \frac{\ell_1}{r} - \frac{3}{2c_V^2 r}(1 + 2\ln r) + \cdots \tag{3.6}$$

where ℓ_1 is a constant that can be computed numerically in terms of L_0 and V_0 (we will not need its value). In writing (3.6) we have chosen a convenient rescaling of the ρ coordinate, so that the conformal boundary metric is

$$g^{(0)}_{\mu\nu} dx^\mu dx^\nu = dx^+ dx^- + \frac{c_V}{4} dx^i dx^i \tag{3.7}$$

As $r \to 0$ the solution approaches $\mathrm{AdS}_3 \times R^2$, with asymptotics

$$\begin{aligned} L_0(r) &= 2br + \mathcal{O}(r^{1+\sigma}) \qquad\qquad \sigma = -\frac{1}{2} + \frac{\sqrt{57}}{6} \\ e^{2V_0(r)} &= 1 + 2r^\sigma + \mathcal{O}(r^{2\sigma}) \end{aligned} \tag{3.8}$$

The AdS_3 radius is

$$L_3 = \frac{1}{b} = \frac{1}{\sqrt{3}} \tag{3.9}$$

Given the AdS_3 factor, we can compute the Brown-Henneaux central charge [13]. If the transverse $x^{1,2}$ space is infinite, this is really a central charge per unit area. If we take the $x^{1,2}$ space to have finite coordinate area V_2, then the central charge is also finite, and given by,

$$c = \frac{3L_3}{2G_3} = \frac{3L_3 V_2}{2G_5} = \frac{\sqrt{3} V_2}{2G_5} \tag{3.10}$$

3.3. CFT interpretation

The above solution describes an RG flow geometry interpolating between AdS_5 at large r and $\mathrm{AdS}_3 \times R^2$ at small r. The CFT interpretation of this flow was given in [1]. In the CFT we have massless charged fermions and bosons propagating in a background magnetic field. In the free field limit, the eigenfunctions are Landau levels, localized in

the $x^{1,2}$ space, but with an arbitrary momentum p_3 parallel to the magnetic field. Aside from the p_3 dependence, the Landau levels have a discrete energy spectrum, and at low energies only the lowest Landau level is occupied, leaving p_3 as the only remaining quantum number. The low energy theory thus reduces to an effective D=1+1 CFT, corresponding to propagation parallel to the magnetic field lines. This explains the appearance of an AdS$_3$ factor in the IR region of the bulk geometry. As on the gravity side, before compactifying the $x^{1,2}$ space we really have a collection of D=1+1 CFTs smeared over the space, but if we compactify with area V_2 then we will obtain a bonafide D=1+1 CFT with a central charge proportional to V_2. In [1] this central charge was computed for free $\mathcal{N}=4$ Super-Yang-Mills theory and compared with the Brown-Henneaux central charge (3.10), with the result $c_{\mathcal{N}=4} = \sqrt{\frac{3}{4}} c_{BH}$. The agreement up to a numerical factor is similar to the agreement up to the factor of 3/4 in the low temperature entropy of D3-branes [25]. As in that case, the solutions under consideration are non-supersymmetric, and there is no symmetry protecting the central charge from being renormalized in going from weak to strong coupling.

Another salient point is that the low energy CFT is populated only by fermionic excitations. This is because the lowest Landau level energy for a massless fermion is $E=0$, while for bosons it is $E \sim \sqrt{B}$. The existence of the fermion zero mode is due to the negative Zeeman energy associated with the fermion spin aligning with the magnetic field, which precisely cancels the kinetic energy. Alternatively, this is a consequence of the index theorem for the Dirac operator. As a consequence, at energies small compared to $\sqrt{B}$, only the lowest fermionic Landau level participates, and so the CFT has only fermionic excitations. Of course, since this is a D=1+1 CFT there is not really a sharp distinction between fermions and bosons.

In the remainder of this paper we use the AdS/CFT correspondence to compute CFT correlation functions at strong coupling.

4. Current correlators

To compute the current two-point function we proceed by solving the Maxwell equations with a specified boundary condition $A^{(0)}_\mu(x)$. Given the solution, we read off the induced current[8] $J^\mu(x)$ from (2.7), and then use this result to extract the correlator according to the formula for first order perturbation theory in $A^{(0)}_\nu$,

$$J^\mu(x) = i \int d^4y \sqrt{g^{(0)}} \langle \mathcal{J}^\mu(x) \mathcal{J}^\nu(y) \rangle A^{(0)}_\nu(y) \tag{4.1}$$

This formula derives from the Minkowski signature functional integral, whence the presence of the prefactor of i; the analytic continuation to the Euclidean version of this formula will be derived in Appendix A.

[8] Throughout, we will denote the current operator by $\mathcal{J}^\mu$ and its expectation value by J^μ; and similarly for the stress tensor operator $\mathcal{T}^{\mu\nu}$ and expectation value $T^{\mu\nu}$.

The effective CFT that we wish to probe inhabits the $x^{\pm}$ directions. For this reason, we will only consider correlators of $\mathcal{J}^{\pm}$. Similarly, working in momentum space, we will only consider nonzero momenta $p_{\pm}$, so that in position space we are integrating the current operators over $x^{1,2}$.

The two-point function can be computed by working to first order in $A_{\mu}^{(0)}$, and so we only need to solve the field equations to linear order around the background solution. In general, the gauge field perturbations can mix at linear order with metric perturbations. However, if the linearized gauge fluctuations have polarization restricted to $A_{\pm}$, and have no dependence on $x^{1,2}$, then it is easy to check that there is no mixing. Hence we can consistently set the metric perturbation to zero in this computation, and we just need to solve the linearized Maxwell-Chern-Simons equation.

4.1. Plane wave expansion

We choose the gauge $A_r = 0$, where the background gauge field obeys,

$$dA_0 = b dx^1 \wedge dx^2 \tag{4.2}$$

We consider a plane wave perturbation with fixed momenta $p_{\pm}$ in the $x^{\pm}$ directions, so that the full gauge field takes the form,

$$A = A_0 + a_+(r, p_{\pm}) e^{ipx} dx^+ + a_-(r, p_{\pm}) e^{ipx} dx^- \tag{4.3}$$

Throughout, we will use the following notations,

$$\begin{aligned} px &= p_+ x^+ + p_- x^- \\ p^2 &= p_+ p_- \end{aligned} \tag{4.4}$$

Substituting (4.3) into $d \star F + kF \wedge F = 0$, we keep only terms linear in $a_{\pm}$. The reduced equations take their simplest form if we define the combinations,

$$\begin{aligned} \varepsilon_p &= p_- a_+ + p_+ a_- \\ \varepsilon_m &= p_- a_+ - p_+ a_- \end{aligned} \tag{4.5}$$

Since the background metric is unperturbed in this computation, we will drop the 0 subscript on L_0 and V_0. The reduced field equations are,

$$\begin{aligned} &(Le^{2V} \varepsilon_m')' - \frac{4k^2 b^2}{Le^{2V}} \varepsilon_m - \frac{2e^{2V}}{L^2} p^2 \varepsilon_m = 0 \\ &Le^{2V} \varepsilon_p' - 2kb\varepsilon_m = 0 \end{aligned} \tag{4.6}$$

For general $p_{\pm}$ these equations must be solved numerically. However, we can make analytical progress by focussing on low energy correlators corresponding to the Euclidean region,

$$0 < p^2 \ll 1 \tag{4.7}$$

This is the regime in which we expect to probe the IR D=1+1 CFT.

To proceed, we employ a standard approach in this context, that of a matched asymptotic expansion (see for example [8]). We consider two overlapping regions that together cover the space, a near region and a far region, in each of which we can solve the equations analytically. Under the assumption (4.7) these regions overlap in a parametrically large region, and by matching the asymptotics there, we obtain a solution valid throughout the entire space.

It is useful to note that the functions V and L appearing in the equations (4.6) depend on no free parameters (assuming that we have set $b = \sqrt{3}$), and so the transition between their small r near horizon behavior and large r asymptotic behavior occurs for $r \approx 1$.

4.2. Near region

The near region is defined as $r \ll 1$. In this region we can use the leading small r asymptotics in V and L,

$$e^{2V} = 1 \qquad\qquad L = 2br \tag{4.8}$$

so that the equations (4.6) become

$$\begin{aligned} &r^2 \varepsilon_m'' + r\varepsilon_m' - k^2 \varepsilon_m - \frac{p^2}{12br}\varepsilon_m = 0 \\ &r\varepsilon_p' - k\varepsilon_m = 0 \end{aligned} \tag{4.9}$$

These equations are equivalent to the three-dimensional Maxwell-Chern-Simons equations $d\star F + 2kbF = 0$, with the metric given by AdS$_3$. Since we are assuming that $p^2 > 0$, the solution for ε_m that is smooth at $r = 0$ is given by a modified Bessel function,

$$\varepsilon_m = \frac{2\sin(2\pi k)}{\pi} C K_{2k}\left(\sqrt{\frac{p^2}{b^3 r}}\right) \tag{4.10}$$

The integration constant C has been chosen so as to make the large $r/p^2 \to \infty$ asymptotics simple, namely

$$\varepsilon_m \sim C\left[\frac{\left(\frac{p^2}{4b^3}\right)^{-k}}{\Gamma(1-2k)} r^k - \frac{\left(\frac{p^2}{4b^3}\right)^{k}}{\Gamma(1+2k)} r^{-k}\right] \tag{4.11}$$

Note that the region of large r/p^2 overlaps with the near region $r \ll 1$ because we are assuming $p^2 \ll 1$. The $r/p^2 \to \infty$ asymptotics of ε_p follow from (4.9),

$$\varepsilon_p \sim C\left[\frac{\left(\frac{p^2}{4b^3}\right)^{-k}}{\Gamma(1-2k)} r^k + \frac{\left(\frac{p^2}{4b^3}\right)^{k}}{\Gamma(1+2k)} r^{-k}\right] + 2p_+ p_- \lambda \tag{4.12}$$

where λ is an independent integration constant, chosen in a convenient manner. The $r/p^2 \to \infty$ asymptotics of the original field $a_\pm$ is then readily found,

$$\begin{aligned} a_+ &\sim \frac{C\left(\frac{p^2}{4b^3}\right)^{-k}}{\Gamma(1-2k)\, p_-} r^k + p_+\lambda \\ a_- &\sim \frac{C\left(\frac{p^2}{4b^3}\right)^{k}}{\Gamma(1+2k)\, p_+} r^{-k} + p_-\lambda \end{aligned} \tag{4.13}$$

4.3. Far region

The far region is defined such that we can neglect the momentum dependent term in (4.6). This is valid provided $p^2/r \ll 1$. The equations (4.6) then reduce to the following equations for $a_\pm$

$$\begin{aligned} Le^{2V} a'_+ - 2kba_+ &= -2kbp_+\tilde\lambda \\ Le^{2V} a'_- + 2kba_- &= 2kbp_-\tilde\lambda \end{aligned} \tag{4.14}$$

where $\tilde\lambda$ is an integration constant. We write the solutions as,

$$\begin{aligned} a_+ &= (a_+^{(0)} - p_+\tilde\lambda)e^{2kb\psi(r)} + p_+\tilde\lambda \\ a_- &= (a_-^{(0)} - p_-\tilde\lambda)e^{-2kb\psi(r)} + p_-\tilde\lambda \end{aligned} \tag{4.15}$$

where the function $\psi(r)$ is familiar from [8], and is defined by,

$$\psi(r) = \int_\infty^r \frac{dr'}{L(r')e^{2V(r')}} \tag{4.16}$$

and $a_\pm^{(0)}$ are new integration constants. The asymptotics of ψ may be evaluated in terms of those of L and V, and we find

$$\begin{aligned} r \to 0 \qquad & \psi(r) = \frac{\ln r}{2b} + \psi_0 + \mathcal{O}(r^\sigma) \\ r \to \infty \qquad & \psi(r) = -\frac{1}{2c_V r} + \mathcal{O}(r^{-2}) \end{aligned} \tag{4.17}$$

where ψ_0 is a constant that may be determined numerically to be $\psi_0 \approx 0.2625$. The asymptotics of $a_\pm$ are then found to be

$$\begin{aligned} r \to 0 \qquad & a_+(r) = (a_+^{(0)} - p_+\tilde\lambda)e^{2kb\psi_0} r^k + p_+\tilde\lambda \\ & a_-(r) = (a_-^{(0)} - p_-\tilde\lambda)e^{-2kb\psi_0} r^{-k} + p_-\tilde\lambda \\ r \to \infty \qquad & a_+(r) = a_+^{(0)} + \frac{1}{4r} a_+^{(2)} \\ & a_-(r) = a_-^{(0)} + \frac{1}{4r} a_-^{(2)} \end{aligned} \tag{4.18}$$

where,

$$a_+^{(2)} = -\frac{4kb}{c_V}(a_+^{(0)} - p_+\tilde{\lambda})$$
$$a_-^{(2)} = \frac{4kb}{c_V}(a_-^{(0)} - p_-\tilde{\lambda}) \tag{4.19}$$

Using the asymptotic relation $\rho = 4r$ of (3.6), we see that the asymptotic expansion of (4.18) agrees with the expansion appearing in (2.5), so we can use (2.7) to read off (the Fourier transform of) the current as,

$$J_\mu = \frac{1}{2\pi G_5} a_\mu^{(2)} \tag{4.20}$$

4.4. Matching

The matching region is defined by $p^2 \ll r \ll 1$, which overlaps both the near and far regions. In this region we demand agreement between the $r/p^2 \to \infty$ asymptotics of the near region solution, and the $r \ll 1$ asymptotics of the far region solution. Equating the expressions in (4.13) with those in the top two lines of (4.18) we find that $\tilde{\lambda} = \lambda$, along with two equations determining C and λ in terms of $a_\pm^{(0)}$. A bit of algebra then leads to the following expressions for $a_\pm^{(2)}$:

$$a_+^{(2)} = -\frac{4kb}{c_V}\frac{a_+^{(0)}}{[1-\zeta\, p^{4k}]} + \frac{4kb}{c_V}\frac{p_+}{p_-}\frac{a_-^{(0)}}{[1-\zeta\, p^{4k}]}$$
$$a_-^{(2)} = +\frac{4kb}{c_V}\frac{a_-^{(0)}}{[1-\zeta^{-1}\, p^{-4k}]} - \frac{4kb}{c_V}\frac{p_-}{p_+}\frac{a_+^{(0)}}{[1-\zeta^{-1}\, p^{-4k}]} \tag{4.21}$$

where

$$\zeta = \zeta(k) = \frac{\Gamma(1-2k)}{\Gamma(1+2k)}\frac{e^{4kb\psi_0}}{(4b^3)^{2k}} \tag{4.22}$$

is a function of k, and of the (fixed) characteristics of the background solution. Note that, extending (4.22) to negative values of its argument, we have $\zeta^{-1} = \zeta(-k)$.

4.5. Correlation functions: IR behavior

Since we are restricting to zero momentum along $x^{1,2}$, it is convenient to define a two-dimensional current, $\hat{\mathcal{J}}_\pm$, by integrating over $x^{1,2}$,

$$\hat{\mathcal{J}}_\pm = \frac{1}{4} c_V V_2 \mathcal{J}_\pm \tag{4.23}$$

Using (4.20) it can be expressed as

$$\hat{\mathcal{J}}_\pm = \frac{c_V c}{4\pi b} a_\pm^{(2)} \tag{4.24}$$

where c is the Brown-Henneaux central charge (3.10).

Using the analytic continuation formulas derived in Appendix A, we read off the correlators from (4.21), expressed in terms of Euclidean momenta,

$$
\begin{aligned}
\langle \hat{\mathcal{J}}_+(p)\hat{\mathcal{J}}_+(-p)\rangle &= \frac{kc}{2\pi}\frac{p_+}{p_-} + \frac{kc}{2\pi}\frac{p_+}{p_-}\left(\frac{\zeta\, p^{4k}}{1-\zeta\, p^{4k}}\right) \\
\langle \hat{\mathcal{J}}_-(p)\hat{\mathcal{J}}_-(-p)\rangle &= \frac{kc}{2\pi}\frac{p_-}{p_+}\left(\frac{\zeta\, p^{4k}}{1-\zeta\, p^{4k}}\right) \\
\langle \hat{\mathcal{J}}_+(p)\hat{\mathcal{J}}_-(-p)\rangle &= -\frac{kc}{2\pi}\left(\frac{\zeta\, p^{4k}}{1-\zeta\, p^{4k}}\right)
\end{aligned}
\tag{4.25}
$$

In writing $\langle \hat{\mathcal{J}}_+\hat{\mathcal{J}}_-\rangle$ we used the freedom to add a contact term, and for the same reason, the correlator can be extracted from either the top or bottom line of (4.21), with the same result modulo contact terms.

The leading non-analytic long-distance behavior, corresponding to small momenta $\zeta p^{4k} \ll 1$ is given by,

$$
\langle \hat{\mathcal{J}}_+(p)\hat{\mathcal{J}}_+(-p)\rangle = \frac{kc}{2\pi}\frac{p_+}{p_-} + \cdots \tag{4.26}
$$

Fourier transforming this to position space gives (see Appendix B for formulas),

$$
\langle \hat{\mathcal{J}}_+(x^+)\hat{\mathcal{J}}_+(y^+)\rangle = -\frac{kc}{2\pi^2}\frac{1}{(x^+-y^+)^2} + \cdots \tag{4.27}
$$

With the conventions adopted here, the minus sign here is actually the one required by unitarity, as confirmed in Appendix C. Keeping the full momentum dependence in (4.25) instead leads to an expansion,

$$
\langle \hat{\mathcal{J}}_+(x)\hat{\mathcal{J}}_+(0)\rangle = -\frac{kc}{2\pi^2}\frac{1}{(x^+)^2}\left(1+\sum_{n=1}^{\infty}\frac{a_n}{(x^+x^-)^{2nk}}\right) \tag{4.28}
$$

where we have used translation invariance to set $y=0$. Similarly, the remaining correlation functions have the structure,

$$
\begin{aligned}
\langle \hat{\mathcal{J}}_-(x)\hat{\mathcal{J}}_-(0)\rangle &= \frac{kc}{2\pi^2}\frac{1}{(x^-)^2}\sum_{n=1}^{\infty}\frac{b_n}{(x^+x^-)^{2nk}} \\
\langle \hat{\mathcal{J}}_+(x)\hat{\mathcal{J}}_-(0)\rangle &= \frac{kc}{2\pi^2}\sum_{n=1}^{\infty}\frac{c_n}{(x^+x^-)^{2nk}}
\end{aligned}
\tag{4.29}
$$

In our derivation of these results we proceeded under the condition that $0 < p^2 \ll 1$. On the one hand, we have neglected corrections of the type p^2/r in the region $r > 1$. On the other hand, in (4.25) we wrote the full functional dependence on p^2. We now make a

comment regarding in what sense this is meaningful. We expect that the full answer, not assuming $p^2 \ll 1$, would add to the expressions in (4.25) terms suppressed by p^2 compared to the leading terms. Since the sub-leading terms in (4.25) are down by powers of p^{4nk}, it is clear that for any fixed k we should only keep a finite number of terms in the expansion, namely those for which $2nk < 1$. By the same token, by taking k sufficiently small we can always arrange for an arbitrarily large number of terms in the expansion to dominate the terms we have been neglecting, and so in this sense the full expressions in (4.25) are meaningful.

4.6. Correlation functions: UV behavior

The UV behavior of the correlators is obtained by solving equations (4.6) for $p_+p_- \gg 1$. Although the short-distance behavior is not central to the main theme of this paper, we shall include it briefly here for the sake of completeness, and verification of overall signs. The UV limit of the correlators reduces to those of pure AdS$_5$. We can therefore set $k = 0$, $e^{2V} = c_V r$, and $L = 2r$ in (4.6). The solution for ε_m which is smooth as $r \to 0$ is proportional to the Bessel function $K_1(2p/\sqrt{r})$. Extracting its large r behavior, we obtain the following expression for the (non-local part of the) current,

$$4\pi G_5 J_\pm = -p_+^2 (\ln p^2) a_\mp^{(0)} \tag{4.30}$$

The UV limits of the current correlators are then found as follows (see Appendices A and B for their evaluation),

$$\begin{aligned} \langle \mathcal{J}_\pm(p)\mathcal{J}_\pm(-p)\rangle &= -\frac{p_\pm^2 \ln p^2}{4\pi G_5} \\ \langle \mathcal{J}_\pm(x)\mathcal{J}_\pm(0)\rangle &= -\frac{1}{2\pi^2 G_5}\frac{1}{(x^\pm)^2(x^+x^-)} \end{aligned} \tag{4.31}$$

Although $\mathcal{J}_\pm$ are dimension 3 operators in the UV, the correlators in (4.31) have a $1/x^4$ falloff because they have been integrated over $x^{1,2}$. Comparing the signs of the position space correlators in the IR of (4.27), and in the UV of (4.31), we find agreement between the signs of the leading coefficients, as is required by unitarity of the low energy sector, and of the full UV theory. Thus, there is no sign change generated by the RG flow.

5. Discussion of current correlators

In this section, we discuss various physical aspects of our results for the current correlation functions.

5.1. Chiral anomaly and unitarity

The chiral anomaly equation for the current $\hat{\mathcal{J}}_\pm$ may be deduced directly from (4.21) in momentum space, and is given by,

$$p_+\hat{\mathcal{J}}_- + p_-\hat{\mathcal{J}}_+ = \frac{kc}{\pi}\left(p_+a_-^{(0)} - p_-a_+^{(0)}\right) \tag{5.1}$$

The anomaly is free of higher order p^{4k} momentum corrections, as expected. The leading IR behavior of the current correlators (4.28) and (4.29) is constrained by the chiral anomaly (5.1). In the bulk, this anomaly equation in the IR arises due to the fact that the D=4+1 Chern-Simons terms $k\int A\wedge F\wedge F$ reduces to a D=2+1 dimensional Chern-Simons term $kb\int A\wedge F$. Bulk Chern-Simons terms are directly related to anomalies of boundary currents. This is manifest here through the fact that the anomaly in (5.1) is linear in k, with parity reversing the sign of k and the roles of the $\pm$ chiralities. It is also manifest in the structure of (4.21) in which chiralities are reversed under $k\to -k$ upon using the fact that $\zeta(-k)=\zeta(k)^{-1}$. Therefore, without loss of generality, we continue to make the choice $k>0$, the case of $k<0$ being obtained by reversing chiralities.

We now use our explicit results of (4.21) and of the correlators (4.25) to analyze the interplay between the chiral anomaly and unitarity of the IR effective CFT. To leading order in the IR, the two-point functions $\langle\hat{\mathcal{J}}_-\hat{\mathcal{J}}_-\rangle$ and $\langle\hat{\mathcal{J}}_+\hat{\mathcal{J}}_-\rangle$ are subdominant, so that $\langle\hat{\mathcal{J}}_+\hat{\mathcal{J}}_+\rangle$ must saturate the chiral anomaly by itself. This forces $\langle\hat{\mathcal{J}}_+\hat{\mathcal{J}}_+\rangle$ in (4.27) to have a $1/(x^+)^2$ falloff with a specified coefficient, $-kc/(2\pi^2)<0$, the sign of which is mandated by unitarity.

Now consider the regime $\zeta p^{4k}\gg 1$ in (4.25). To the extent that (4.25) represent the correlators of some D=1+1 QFT, this limit probes the short distance regime of this QFT. One sees that the situation as regards the chiral anomaly is now reversed: the correlators $\langle\hat{\mathcal{J}}_+\hat{\mathcal{J}}_+\rangle$ and $\langle\hat{\mathcal{J}}_+\hat{\mathcal{J}}_-\rangle$ are now subdominant, and it is the correlator $\langle\hat{\mathcal{J}}_-\hat{\mathcal{J}}_-\rangle$ which saturates the anomaly. The coefficient of $1/(x^-)^2$ in this correlator is now given by $kc/(2\pi^2)$, and is opposite to the one required by "naive unitarity" in this regime. This is telling us that the expressions in (4.25) cannot by themselves be interpreted as the correlators of some unitary QFT; there must be corrections that set in in the UV to maintain unitarity. In the present context we know precisely what these corrections represent. We have already noted that the approximations leading to (4.25) break down for $p\sim 1$. For momenta larger than this, the correlators will start to "see" the AdS$_5$ region. Unitarity in the full magnetic brane solution with AdS$_5$ asymptotics is maintained, as may be seen explicitly, for example, from the UV limit of the correlators given in (4.31). These facts fit together nicely.

5.2. Sub-leading terms and double trace operators

Next, let us consider the interpretation of the sub-leading terms. In the near horizon geometry that governs the IR physics, the bulk Maxwell field obeys a D=2+1 dimensional

Maxwell-Chern-Simons equation. As is well known, the Chern-Simons term gives the gauge field a mass proportional to k. On the boundary, this corresponds to an operator with a k-dependent scaling dimension, which accounts for the k dependent powers appearing in the correlation functions. On the other hand, the fact that we have an infinite series of different powers appearing in the correlators indicates that these operators do not have a definite scaling dimension, and that the theory is not scale invariant. The reason for this can be explained in terms of modified boundary conditions and double trace operators.

To understand this let us examine the relation between the behavior of the gauge field near the UV AdS_5 boundary versus at the near horizon IR AdS_3 boundary. In particular, we focus on the relation between the "source" and "vev" terms in the two regions. At the AdS_5 boundary we are using the standard AdS/CFT dictionary, which identifies the source as $a_\pm^{(0)}$ and the vev as $a_\pm^{(2)}$. Now, at the boundary of the AdS_3 region, which corresponds to the matching region $|p_+p_-| \ll r \ll 1$, a generic solution of the field equations has the expansion

$$\begin{aligned} a_+ &\sim C_+ r^k + p_+\lambda \\ a_- &\sim C_- r^{-k} + p_-\lambda \end{aligned} \tag{5.2}$$

In terms of these quantities, it is not immediately obvious how to relate source and vev terms to C_+, C_- and λ. However, since the field equations in the far region relate these coefficients to the data at the AdS_5 boundary, we can use the AdS_5/CFT_4 dictionary to answer this question.

First consider the current. From (4.18)-(4.21), we see that there is a simple relation $\mathcal{J}_\pm \propto C_\pm$, between the current and the coefficients of the $r^{\pm k}$ terms in the matching region. Thus the current can be immediately read off from the AdS_3 near-boundary behavior shown in (5.2).

For the source, comparing (4.18) to (5.2) now leads to the relations

$$\begin{aligned} a_+^{(0)} &= e^{-2kb\psi_0} C_+ + p_+\lambda \\ a_-^{(0)} &= e^{2kb\psi_0} C_- + p_-\lambda \end{aligned} \tag{5.3}$$

We can think of this as a version of mixed Neumann-Dirichlet boundary conditions at the AdS_3 boundary. Note that all the information about the far region is contained in the factor $e^{2kb\psi_0}$; if we imagine a more general family of geometries interpolating between the AdS_3 and AdS_5 regions we can think of $e^{2kb\psi_0}$ as being a variable parameter that controls the form of the IR boundary condition.

In AdS/CFT it is well known that considering mixed boundary conditions corresponds to adding double trace interactions to the Lagrangian of the boundary CFT [14]. These double trace interactions induce a nontrivial RG flow in the field theory, and so correlation functions acquire a nontrivial dependence on momenta, exhibiting the interpolation between the UV and IR fixed points. The explicit form of such correlators is determined using large N factorization, and takes the form shown in (4.25).

The fact that we generate double trace interactions in the approach to the IR fixed point is not surprising. We start from the UV CFT at the AdS_5 boundary, and then add an external magnetic field that introduces a scale. The theory then undergoes an RG flow to the IR, generating in the process all possible operators allowed by symmetry and large N counting. The appearance of double trace interactions in holographic RG flows has been discussed in detail in recent papers [11][12], and a discussion of boundary conditions for gauge fields in AdS is found in [26].

We now consider in more detail the form of the double trace interactions. To do so, we first define the CFT in the absence of double trace terms. We write the current operators in the IR CFT as

$$\begin{aligned} \hat{\mathcal{J}}_+ &= \sqrt{\frac{kc}{2\pi}}\,\partial_+\phi + \sqrt{\frac{kc\zeta}{2\pi}}\partial_+\mathcal{O} \\ \hat{\mathcal{J}}_- &= \sqrt{\frac{kc\zeta}{2\pi}}\partial_-\mathcal{O} \end{aligned} \tag{5.4}$$

Here ϕ is a free boson, whose momentum space two-point function is,

$$\langle\phi(p)\phi(-p)\rangle = \frac{1}{p^2} \tag{5.5}$$

while $\mathcal{O}$ is a scalar operator of dimension (k,k), with two-point function,

$$\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle = (p^2)^{2k-1} \tag{5.6}$$

The mixed correlator $\langle\phi\mathcal{O}\rangle$ is assumed to vanish. Given these two-point functions, if we now add to the CFT Lagrangian the double trace term $\zeta\partial_+\mathcal{O}\partial_-\mathcal{O}$, and use large N factorization, it is easy to see that we recover the correlators displayed in (4.25). On the one hand, since $\partial_+\mathcal{O}\partial_-\mathcal{O}$ is an operator of total scaling dimension $2k+2$, it is irrelevant in the RG sense for any $k>0$. On the other hand, the total scaling dimension $2k$ of the operator $\mathcal{O}$ itself is below the Breitenlohner-Freedman bound (whose value is 1 for the asymptotic AdS_3 near-horizon region) when $0<k<1/2$, thus suggesting the existence of an instability in this range of k. Remarkably, the same range of k is singled out in the presence of non-zero charge density in [17].

To summarize, the content of the IR CFT in the sector dual to the bulk gauge field consists of a boson ϕ together with the operator $\mathcal{O}$ of dimension (k,k). The theory contains a double trace interaction $\zeta\partial_+\mathcal{O}\partial_-\mathcal{O}$. From the point of view of the IR theory, ζ can be viewed as free parameter. It takes a definite value upon embedding the theory in a specific UV CFT, as in (4.22).

5.3. The $k\to 0$ limit of the sub-leading terms

In the limit $k\to 0$, the contribution of the Chern-Simons term in the action vanishes, and the chiral anomaly is cancelled. But the structure of the current correlators is non-trivial. In fact, in the small k limit, it becomes reliable to keep all the sub-leading expansion

terms in the current correlators (4.25) since further corrections in integer powers of p^2 will now be small compared to all expansion terms, as long as $p^2 \ll 1$. The $k \to 0$ limits of the current correlators are as follows,

$$
\begin{aligned}
\langle \hat{\mathcal{J}}_+(p)\hat{\mathcal{J}}_+(-p)\rangle &= -\frac{c\,p_+}{2\pi p_-}\,\frac{1}{\zeta'(0)+2\ln(p^2)} \\
\langle \hat{\mathcal{J}}_-(p)\hat{\mathcal{J}}_-(-p)\rangle &= -\frac{c\,p_-}{2\pi p_+}\,\frac{1}{\zeta'(0)+2\ln(p^2)} \\
\langle \hat{\mathcal{J}}_+(p)\hat{\mathcal{J}}_-(-p)\rangle &= \frac{c}{2\pi}\,\frac{1}{\zeta'(0)+2\ln(p^2)}
\end{aligned}
\tag{5.7}
$$

where $\zeta'(0$ is the derivative in k at $k = 0$ of the function $\zeta(k)$ of (4.22). To leading order in small p^2, the $\zeta'(0)$ term may in fact be omitted. A pole appears at a finite value of p^2 in (5.7), but this effect is of course beyond the range of validity of (5.7).

5.4. Relation to Luttinger liquid theory

Before turning to stress tensor correlators, let us mention also the connection between our results and those appearing in the Luttinger liquid approach to interacting condensed matter systems in D=1+1 (see, e.g., [27].) Such systems are studied using bosonization methods. Linearizing around the Fermi surface, one obtains fermions with a relativistic type dispersion relation, and these can be bosonized in a standard fashion. In the simplest setup corresponding to spinless fermions interacting via four-fermi terms, the charge density operator is bosonized as

$$
\rho \sim \partial_x\phi + \left[e^{2ik_F x+2i\phi} + \text{H.C.}\right]
\tag{5.8}
$$

where ϕ is a free compact boson, whose radius depends on the four-Fermi couplings. We can compare this with the density operator $\rho = \hat{\mathcal{J}}_+ + \hat{\mathcal{J}}_-$ obtained from (5.4). In both cases, the addition of the extra operators, beyond the ϕ derivative piece, leads to subleading terms in the density-density correlators. One obvious difference between (5.4) and (5.8) is the presence of the $e^{2ik_F x}$ factor in (5.8), which corresponds to a process where a fermion is taken from one side of the Fermi surface to the other. There is no analog of such a factor in our case because we are working at zero density. Correlators at finite density, and their connections with Luttinger liquids, will be presented in [17]. See [28][29][30] for other discussions of holography and Luttinger liquids.

6. Stress tensor correlators

The strategy for computing stress tensor two-point functions is very similar to that employed in the last section for the current correlators. We solve the linearized Einstein

equations with specified boundary condition $g^{(0)}_{\mu\nu}$. Given this solution we compute $T^{\mu\nu}(x)$, and then use this to read off the two-point function via,

$$T^{\mu\nu}(x) = \frac{i}{2}\int d^4y\sqrt{g^{(0)}}\langle \mathcal{T}^{\mu\nu}(x)\mathcal{T}^{\alpha\beta}(y)\rangle g^{(0)}_{\alpha\beta}(y) \tag{6.1}$$

To isolate the effective D=1+1 CFT, we restrict the graviton polarizations to the $x^{\pm}$ components, and restrict the momenta to $p_{\pm}$. In the regime $0 < p_+p_- \ll 1$ we can solve the problem analytically by using a matched asymptotic expansion.

More precisely, we consider the field configuration

$$\begin{aligned} ds^2 &= \frac{dr^2}{L^2} + 2Ldx^+dx^- + M(dx^+)^2 + N(dx^-)^2 + e^{2V_0}dx^idx^i \\ F &= bdx^1 \wedge dx^2 \end{aligned} \tag{6.2}$$

with

$$\begin{aligned} L(r) &= L_0(r) + L_1(r)e^{ipx} \\ M &= M_1(r)e^{ipx} \\ N &= N_1(r)e^{ipx} \end{aligned} \tag{6.3}$$

The perturbations are therefore given by L_1, M_1, and N_1, and we proceed by solving the Einstein equations to linear order in these functions. In writing (6.2) we have used the fact that with this choice of metric perturbation there is no mixing with gauge field fluctuations, and so the latter can be set to zero. We have also made a choice of coordinates which is convenient for the analysis of the far region equations.

The perturbation L_1 will be set to zero when we come to the far region, although it is helpful to retain it for the time being. It is only necessary to consider the L_1 perturbation if we are interested in computing a two-point function involving T_{+-}. However, all such correlators are pure contact terms, vanishing when the two operators are at distinct points. This is a reflection of the trace anomaly of the IR D=1+1 CFT, stating that T_{+-} can be expressed locally in terms of the conformal boundary metric; namely, it is just proportional to the Ricci scalar. For this reason, up to contact terms, all low energy correlators can be accessed while setting $L_1 = 0$.

6.1. Near region

As in the last section, the near region is defined by $r \ll 1$, where we can set $L_0 = 2br$ and $e^{2V_0} = 1$. It is instructive to solve the linearized Einstein equations in a two-step process. We first solve the equations in a different coordinate system than in (6.2), and then perform the appropriate coordinate transformation to put the solution in the form of (6.2). In particular, we first consider the perturbed field configuration

$$\begin{aligned} ds^2 &= ds^2_B + \Big[h_{++}(r)(dx^+)^2 + 2h_{+-}(r)dx^+dx^- + h_{--}(r)(dx^-)^2\Big]e^{ipx} \\ F &= bdx^1 \wedge dx^2 \end{aligned} \tag{6.4}$$

where ds_B^2 now denotes the background metric (3.1). The general solution of the linearized Einstein equations, given the *Ansatz* (6.2), is

$$\begin{aligned} h_{++}(r) &= s_{++}r + t_{++} \\ h_{--}(r) &= s_{--}r + t_{--} \\ h_{+-}(r) &= s_{+-}r + t_{+-} \end{aligned} \tag{6.5}$$

where s and t are independent of r with,

$$\begin{aligned} t_{+-} &= \frac{1}{24b}\left(p_+^2 s_{--} + p_-^2 s_{++} - 2p_+p_-s_{+-}\right) \\ p_-t_{++} &= p_+t_{+-} \\ p_+t_{--} &= p_-t_{+-} \end{aligned} \tag{6.6}$$

The structure of the solution is easy to understand. The equations governing $h_{\mu\nu}$ are just those of D=2+1 gravity expanded around AdS_3, and solutions are thus locally pure gauge. Furthermore, the coefficients $s_{\mu\nu}$ and $t_{\mu\nu}$ can be identified in terms of the AdS_3/CFT_2 correspondence: $s_{\mu\nu}$ represents the perturbation of the conformal boundary metric, and $t_{\mu\nu}$ is proportional to the boundary stress tensor. The top line of (6.6) is then identified as the linearized trace anomaly for the stress tensor (the right hand side is proportional to the linearized Ricci scalar), while the bottom two lines are the equations representing conservation of the stress tensor.

We now change coordinates to put the perturbed solution of (6.5) and (6.6) in the form of (6.2). In fact, for what follows we only need the resulting solution in the region $p^2 \ll r \ll 1$, for which we find

$$\begin{aligned} L_1(r) &\sim t_{+-} \\ M_1(r) &\sim s_{++}r + t_{++} \\ N_1(r) &\sim s_{--}r + t_{--} \end{aligned} \tag{6.7}$$

with

$$\begin{aligned} t_{++} &= \frac{1}{24b}\left(p_+p_-s_{++} + \frac{p_+^3}{p_-}s_{--}\right) \\ t_{--} &= \frac{1}{24b}\left(p_+p_-s_{--} + \frac{p_-^3}{p_+}s_{++}\right) \end{aligned} \tag{6.8}$$

The relations (6.8) follow from (6.6) with $s_{+-} = 0$. All dependence on s_{+-} is absorbed by the coordinate transformation. Note that t_{+-} in (6.7) is arbitrary; this is a reflection of the freedom to shift the radial coordinate r while preserving the gauge choice (6.2).

6.2. Far region

To obtain the far region equations we assume $r \gg p^2$, so that momentum dependent terms in the field equations can be dropped. We will now take

$$L_1 = 0 \tag{6.9}$$

This is required by the field equations once we demand that there be no perturbation of the field V. As explained above, setting $L_1 = 0$ is permitted when computing the non-contact part of the correlators.

Substituting (6.2) into the linearized Einstein equation, and dropping the momentum dependent terms, we find the following equations

$$\begin{aligned} M_1'' + 2V_0' M_1' + 4(V_0'' + V_0'^2) M_1 &= 0 \\ N_1'' + 2V_0' N_1' + 4(V_0'' + V_0'^2) N_1 &= 0 \end{aligned} \tag{6.10}$$

We now note that M_1 and N_1 obey the same equation as obeyed by L_0 in the top line of (3.3). Thus, one solution is given by L_0 itself, which is easily seen to correspond to acting on the background solution with a coordinate transformation of the form $x^+ \to x^+ + \epsilon x^-$, or the equivalent with $x^+ \leftrightarrow x^-$. The second linearly independent solution was obtained in [8] (see section 5.4 of that paper) and denoted by L_0^c. It is given explicitly by,

$$L_0^c(r) = L_0(r) \int_\infty^r \frac{dr'}{L_0(r')^2 e^{2V_0(r')}} \tag{6.11}$$

This function has the following asymptotics,

$$\begin{aligned} &r \to 0 \qquad && L_0^c \sim -\frac{1}{2b} \\ &r \to \infty \qquad && L_0^c \sim -\frac{1}{4c_V r} \end{aligned} \tag{6.12}$$

The solutions for M_1 and N_1 are arbitrary linear combinations of L_0 and L_0^c,

$$\begin{aligned} M_1(r) &= 2L_0(r) g^{(0)}_{++} - c_V L_0^c(r) \tilde{g}^{(4)}_{++} \\ N_1(r) &= 2L_0(r) g^{(0)}_{--} - c_V L_0^c(r) \tilde{g}^{(4)}_{--} \end{aligned} \tag{6.13}$$

We have labelled the coefficients in a convenient manner, noting that the $r \to \infty$ asymptotics are given by,

$$\begin{aligned} M_1(r) &\sim (4r + \cdots)\, g^{(0)}_{++} + \frac{1}{4r} \tilde{g}^{(4)}_{++} \\ N_1(r) &\sim (4r + \cdots)\, g^{(0)}_{--} + \frac{1}{4r} \tilde{g}^{(4)}_{--} \end{aligned} \tag{6.14}$$

In particular, this identifies $g^{(0)}_{\mu\nu}$ as the conformal boundary metric appearing in (2.4). On the other hand, $\tilde{g}^{(4)}_{\mu\nu}$ is not quite the same as $g^{(4)}_{\mu\nu}$ appearing in (2.4). The discrepancy arises from the contribution of $1/r$ terms appearing in the $\cdots$ terms in (6.14). It is clear that $\tilde{g}^{(4)}_{\mu\nu}$ and $g^{(4)}_{\mu\nu}$ differ by an amount proportional to $g^{(0)}_{\mu\nu}$. But, as noted in the discussion after (2.7), we are not keeping track of contributions to $g^{(4)}_{\mu\nu}$ that are local in $g^{(0)}_{\mu\nu}$ anyway, since these only show up in the correlators as contact terms. From now on, we therefore ignore the distinction, and simply write $\tilde{g}^{(4)}_{\mu\nu} = g^{(4)}_{\mu\nu}$.

The small r asymptotics are

$$
p^2 \ll r \ll 1 \qquad \begin{aligned} M_1(r) &\sim 4brg^{(0)}_{++} + \frac{c_V}{2b} g^{(4)}_{++} \\ N_1(r) &\sim 4brg^{(0)}_{--} + \frac{c_V}{2b} g^{(4)}_{--} \end{aligned} \tag{6.15}
$$

6.3. Matching

Matching the expansions (6.7) and (6.15) in the overlap region $p^2 \ll r \ll 1$ gives,

$$
\begin{aligned}
s_{++} &= 4bg^{(0)}_{++} \\
s_{--} &= 4bg^{(0)}_{--} \\
t_{++} &= \frac{c_V}{2b} g^{(4)}_{++} \\
t_{--} &= \frac{c_V}{2b} g^{(4)}_{--} \\
t_{+-} &= 0
\end{aligned} \tag{6.16}
$$

along with the relations (6.8), which can now be written as

$$
\begin{aligned}
g^{(4)}_{++} &= \frac{b}{3c_V} \frac{p_+^3}{p_-} g^{(0)}_{--} + \frac{b}{3c_V} p_+ p_- g^{(0)}_{++} \\
g^{(4)}_{--} &= \frac{b}{3c_V} \frac{p_-^3}{p_+} g^{(0)}_{++} + \frac{b}{3c_V} p_+ p_- g^{(0)}_{--}
\end{aligned} \tag{6.17}
$$

We note that the last term in each line is analytic in momentum, and hence local in position space, and thus only contributes to contact terms in correlators.

6.4. Correlation functions: IR behavior

In analogy to what was done in (4.23) for the current, we work in terms of the two-dimensional stress tensor defined as

$$
\hat{\mathcal{T}}_{\pm\pm} = \frac{1}{4} c_V V_2 \mathcal{T}_{\pm\pm} \tag{6.18}
$$

with a corresponding relation for the expectation value of this operator. Using (2.7), (3.10), and (6.17), we obtain

$$\begin{aligned} \hat{T}_{++} &= \frac{c}{24\pi}\frac{p_+^3}{p_-}g^{(0)}_{--} + \text{local} \\ \hat{T}_{--} &= \frac{c}{24\pi}\frac{p_-^3}{p_+}g^{(0)}_{++} + \text{local} \\ \hat{T}_{+-} &= 0 + \text{local} \end{aligned} \tag{6.19}$$

We can now use (6.1) to read off the momentum space correlators (note the factors of 2 appearing when lowering indices using $g^{(0)}_{+-} = \frac{1}{2}$)

$$\begin{aligned} \langle \hat{\mathcal{T}}_{++}(p)\hat{\mathcal{T}}_{++}(-p)\rangle &= \frac{c}{48\pi}\frac{p_+^3}{p_-} \\ \langle \hat{\mathcal{T}}_{--}(p)\hat{\mathcal{T}}_{--}(-p)\rangle &= \frac{c}{48\pi}\frac{p_-^3}{p_+} \end{aligned} \tag{6.20}$$

with all other correlators vanishing (up to contact terms). Fourier transforming to position space gives

$$\begin{aligned} \langle \hat{\mathcal{T}}_{++}(x)\hat{\mathcal{T}}_{++}(0)\rangle &= \frac{c}{8\pi^2}\frac{1}{(x^+)^4} \\ \langle \hat{\mathcal{T}}_{--}(x)\hat{\mathcal{T}}_{--}(0)\rangle &= \frac{c}{8\pi^2}\frac{1}{(x^-)^4} \end{aligned} \tag{6.21}$$

These are the standard formulas for the correlation functions of the stress tensor in a D=1+1 CFT, with c being the central charge. In particular, this demonstrates that the central charge appearing in the IR CFT matches the Brown-Henneaux central charge defined in the near horizon AdS_3 region.

6.5. Correlation functions: UV behavior

The UV behavior of the stress tensor correlators may be computed by carrying out perturbation theory around pure AdS_5. The calculations are similar to those of the current correlators, and will not be given in detail here. The metric fluctuations now involve the Bessel function $K_2(2p/\sqrt{r})$. From its $r \to \infty$ behavior, we read off the non-local contributions,

$$4\pi G_5 T_{\pm\pm} = -\frac{1}{12}p_\pm^4 \ln(p^2) g^{(0)}_{\mp\mp} \tag{6.22}$$

The corresponding leading UV position space correlators are given by,

$$\langle \mathcal{T}_{++}(x)\mathcal{T}_{++}(0)\rangle = \frac{1}{4\pi^2 G_5 (x^+)^4(x^+x^-)} \tag{6.23}$$

Comparing the signs of the position space correlators of (6.21) in the IR, and of (6.23) in the UV, we find agreement, as is expected by unitarity in the IR sector and the full UV theory. RG flow does not reverse this sign.

6.6. Sub-leading contributions in the stress tensor correlators

The stress tensor correlators exhibit an infinite series of sub-leading terms in fractional powers of momenta, just as the current correlators did. Their origin can be traced back to the fluctuation modes in the field V, which change the size of the internal $x^{1,2}$-space. In turn, these modes feed back into the other components of the metric. For simplicity, we shall focus here on the correlator of the stress tensor component $\mathcal{T}_V$ conjugate to V, which is defined by $\mathcal{T}_{ij} = \delta_{ij}\mathcal{T}_V$ for $i, j = 1, 2$. The near region solution for the fluctuations h_{ij} is given by,

$$h_{ij}(r) = \frac{v_0 \delta_{ij}}{\sqrt{r}} \frac{2\sin(2\pi\sigma)}{\pi} K_{2\sigma-1}\left(\sqrt{\frac{p^2}{b^3 r}}\right) \tag{6.24}$$

where v_0 is constant. This formula bears strong resemblance to (4.10) for the current correlators, but with the index $2k$ replaced by $2\sigma - 1$, where σ was given in (3.8), and the parameter $\zeta = \zeta(k)$ of (4.10) replaced by ζ_V. In the overlap region, where $p^2 \ll r \ll 1$, the behavior of the fluctuations h_{ij} simplifies,

$$h_{ij}(r) \sim v_+ r^{\sigma} + v_- r^{-\sigma-1} \tag{6.25}$$

where we have the following formula for the ratio,

$$\frac{v_-}{v_+} = \zeta_V p^{4\sigma+2} \qquad\qquad \zeta_V = -\frac{\Gamma(-2\sigma)}{\Gamma(2+2\sigma)(4b^3)^{2\sigma+1}} \tag{6.26}$$

Extracting the correlator, we find the following structure,

$$\langle \mathcal{T}_V(p)\mathcal{T}_V(-p)\rangle \sim \frac{\zeta_V p^{4\sigma+2}}{1 - \zeta_V p^{4\sigma+2}} \tag{6.27}$$

This corresponding geometric series expansion may again be inferred from the presence of a double trace interaction $\zeta_V \partial_+ \mathcal{O}_V \partial_- \mathcal{O}_V$, this time of a scalar operator $\mathcal{O}_V$ of dimension $(\sigma + 1/2, \sigma + 1/2)$, whose two-point function is given by,

$$\langle \mathcal{O}_V(p)\mathcal{O}_V(-p)\rangle = p^{4\sigma} \tag{6.28}$$

We note that the (total) dimension $4\sigma+4$ of $\partial_+\mathcal{O}_V\partial_-\mathcal{O}_V$ makes this perturbation irrelevant in the IR. The total dimension $2\sigma+1$ of the operator $\mathcal{O}_V$ itself is above the IR Breitenloner-Freedman bound of dimension $D/2 = 1$.

We close by noting that here, just as with the sub-leading contributions to the current correlators, there is also an infinite series of further sub-leading corrections in the form of integer powers of p^2 due to the fact that we have neglected corrections of the type p^2/r in the far region.

6.7. Emergence of IR Virasoro algebra

The preceding computation shows that the correlation functions of the stress tensor defined at the AdS$_5$ boundary are consistent with the existence of an IR D=1+1 CFT corresponding to the near horizon AdS$_3$ factor. Associated with such a CFT is a Virasoro algebra, and it is well known how this arises in AdS$_3$ [13]. Namely, one considers coordinate transformations that preserve the asymptotic AdS$_3$ boundary conditions. Such coordinate transformations include those that act as conformal transformations of the AdS$_3$ boundary coordinates (we will refer to these as Brown-Henneaux coordinate transformations), and the transformation law for the boundary stress tensor under these establishes the existence of a Virasoro algebra [23].

It is interesting to consider how the story is modified when we embed the near horizon AdS$_3$ factor in an asymptotically AdS$_5$ geometry. In particular, the standard boundary conditions defining an asymptotically AdS$_5$ geometry are incompatible with an infinite dimensional group of coordinate transformations acting on the boundary coordinates, but naively it would seem that these are required for the existence of the infinite dimensional Virasoro algebra. This puzzle can be addressed concretely using the interpolating geometry at hand. As we will see, the resolution is that the pure coordinate transformations appearing in the AdS$_3$ region extend to physical modes (i.e. modes that cannot be undone by a coordinate transformation) in the AdS$_5$ region. The behavior of the AdS$_5$ stress tensor under the inclusion of these modes is what gives rise to the Virasoro algebra.

With this motivation, we now construct a perturbed asymptotically AdS$_5$ solution whose near geometry contains an AdS$_3$ factor perturbed by a Brown-Henneaux coordinate transformation. It is easiest to start in the AdS$_3$ region and then extend the solution outwards to the AdS$_5$ region. In our coordinates, the metric of the AdS$_3$ factor is

$$ds^2 = \frac{dr^2}{4b^2r^2} + 4br dx^+ dx^- \tag{6.29}$$

Neither the R^2 factor nor the gauge field will play any role in what follows, and so we suppress them.

Now consider the following infinitesimal Brown-Henneaux coordinate transformation

$$\begin{aligned} r &\to r + \epsilon^r(r) e^{ip_+x^+} \\ x^+ &\to x^+ + \epsilon^+ e^{ip_+x^+} \\ x^- &\to x^- + \epsilon^-(r) e^{ip_+x^+} \end{aligned} \tag{6.30}$$

where ϵ^+ is a constant and

$$\begin{aligned} \epsilon^r(r) &= -ip_+\epsilon^+ r \\ \epsilon^-(r) &= \frac{p_+^2}{8b^3 r}\epsilon^+ \end{aligned} \tag{6.31}$$

To first order in ϵ^+ the metric becomes

$$ds^2 = \frac{dr^2}{4b^2r^2} + 4br dx^+ dx^- + \frac{i}{2b^2} p_+^3 \epsilon^+ e^{ip_+x^+} (dx^+)^2 \tag{6.32}$$

This is a Brown-Henneaux coordinate transformation acting as a reparameterization of x^+; there is also the obvious analogous transformation acting on x^-. In terms of the asymptotic data written in (6.7) this corresponds to

$$t_{++} = \frac{i}{2b^2} p_+^3 \epsilon^+ \tag{6.33}$$

and with $s_{++} = s_{--} = t_{--} = t_{+-} = p_- = 0$.

We can now use formulas (6.15) and (6.16) to construct the full asymptotically AdS_5 solution with this near horizon behavior. We simply take

$$M_1 = -c_V g_{++}^{(4)} L_0^c(r) \tag{6.34}$$

with

$$g_{++}^{(4)} = \frac{2b}{c_V} t_{++} \tag{6.35}$$

As advertised above, the perturbation mode (6.34) is physical, and cannot be undone by a coordinate transformation. Equation (6.35) shows that, up to a proportionality constant, the stress tensor measured at the AdS_5 boundary is equal to the stress tensor measured at the AdS_3 boundary. The Schwarzian derivative transformation law obeyed by the AdS_3 stress tensor will thus be transferred to the AdS_5 stress tensor. We can also see that the proportionality constant is precisely such that it gives the correct central charge in the Schwarzian derivative. In particular, we compute

$$\hat{T}_{++} = \frac{c}{24\pi} i p_+^3 \epsilon^+ \tag{6.36}$$

or in position space,

$$\hat{T}_{++} = \frac{c}{24\pi} \partial_+^3 \epsilon^+ \tag{6.37}$$

which is the Schwarzian derivative term with the correct normalization. Of course, there is also an ordinary tensor transformation part which is absent here since we're starting from a solution with vanishing stress tensor. Finally, interchanging $\pm$ indices in this computation yields the analogous transformation law for $\hat{T}_{--}$. To summarize, the AdS_5 stress tensor inherits the properties of the near horizon AdS_3 stress tensor, yielding a pair of Virasoro algebras with the expected central charges.

7. Conclusion

Our objective was to study how properties of the IR CFT dual to the near horizon $AdS_3 \times R^2$ geometry could be extracted from correlation functions computed at the AdS_5 boundary. We found a consistent picture, after taking into account various subtleties. In particular, to recover the IR Virasoro algebras associated with AdS_3 we had to take into account the fact that Virasoso generators do not implement pure coordinate transformations, as they do in the asymptotically AdS_3 context. And for the current correlators we had to take into account the role played by double trace interactions. A pleasing aspect of these computations was that they could be carried out almost entirely analytically, due to the simplifications occurring in the low energy limit.

In the example studied here, the IR geometry is a familiar one, since it contains an AdS factor. But it is worth noting that the same strategy of deducing the properties of the IR theory from the correlators evaluated at the UV boundary can be employed in cases where the near horizon geometry is not so familiar, and hence where the IR CFT is not known, if it exists at all. An example of this occurs in our setup when we turn on a nonzero charge density. In that case, it was shown in [8] that the near horizon geometry contains a null warped AdS_3 factor. Furthermore, this system undergoes a quantum phase transition at a critical value of the dimensional ratio of the charge density to magnetic field. These facts make it especially interesting to consider the computation of correlation functions in these charged magnetic RG flow geometries. Results of such computations will be presented separately [17].

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Appendix A. Minkowskian, Euclidean, and analytic continuation

We have chosen to define the Einstein-Maxwell-Chern-Simons action, the current, and the stress tensor with respect to the Minkowski signature. The evaluation of correlators is, however, more cleanly carried out with respect to the Euclidean signature. Via analytic continuation and suitable $i\epsilon$ prescriptions we can then obtain the various Minkowski signature correlators (time ordered, retarded, advanced, etc.). In this appendix we provide formulas needed to effect the analytic continuation to obtain the Minkowski signature time ordered correlator from the Euclidean version.

Minkowski time t_M is related to Euclidean time t_E by,

$$t_M = -it_E \tag{A.1}$$

The analytic continuation of the variational formula (2.6), which defines the boundary current and stress tensor in terms of the Minkowski signature action $S = S_M$, is then

given by,

$$i\delta S_M = \int dt_E d^3x \sqrt{g^{(0)}} \left(\frac{1}{2} T^{\mu\nu} \delta g^{(0)}_{\mu\nu} + J^\mu \delta A^{(0)}_\mu \right) \tag{A.2}$$

Note that, with our conventions for the boundary asymptotic metric rdx^+dx^-, the quantity $g^{(0)} = -\det(g^{(0)})$ remains positive through this analytic continuation. Also, it is conventional to define the Euclidean signature action S_E by $-S_E = iS_M$, but this quantity will not be needed here. As a result, the proper analytic continuations of formulas (4.1) and (6.1) are as follows,[9]

$$\begin{aligned} J^\mu(x) &= \int d^4y \sqrt{g^{(0)}} \langle \mathcal{J}^\mu(x) \mathcal{J}^\nu(y) \rangle A^{(0)}_\nu(y) \\ T^{\mu\nu}(x) &= \frac{1}{2} \int d^4y \sqrt{g^{(0)}} \langle \mathcal{T}^{\mu\nu}(x) \mathcal{T}^{\alpha\beta}(y) \rangle g^{(0)}_{\alpha\beta}(y) \end{aligned} \tag{A.3}$$

The space-time coordinates x, y, and the correlators, refer to Euclidean signature. In particular, the Fourier transforms with respect to Euclidean momenta of the current and stress tensor are given as follows,

$$\begin{aligned} J^\mu(p) &= \langle \mathcal{J}^\mu(p) \mathcal{J}^\nu(-p) \rangle A^{(0)}_\nu(p) \\ T^{\mu\nu}(p) &= \frac{1}{2} \langle \mathcal{T}^{\mu\nu}(p) \mathcal{T}^{\alpha\beta}(-p) \rangle g^{(0)}_{\alpha\beta}(p) \end{aligned} \tag{A.4}$$

to first order in $A^{(0)}_\nu(p)$ and $g^{(0)}_{\alpha\beta}(p)$. It is these formulas that were used to extract the current correlators of (4.25) from (4.21), (4.31) from (4.30), and the stress tensor correlators (6.20) from (6.19), all expressed in terms of Euclidean momenta.

Appendix B. Fourier transforms

We need the Fourier transforms of various correlators from momentum space to position space. Let us first note a few basic conventions. The $x^\pm$ part of the metric is written as

$$ds^2 = dx^+ dx^- \tag{B.1}$$

We can view this metric either on Minkowski space for real coordinates $x^\pm = x \pm t$, or on Euclidean space for complex coordinates $x^\pm = \sigma^1 \pm i\sigma^2$, with real $\sigma^{1,2}$. We shall also use the notations, $px = p_+x^+ + p_-x^-$ and $p^2 = p_+p_-$. Various useful Fourier transforms in Eulcidean signature may be deduced from the following basic family of integrals,

$$\int \frac{d^2p}{(2\pi)^2} e^{ipx} \left(p^2\right)^a = \frac{2^{2a}\, \Gamma(1+a)}{\pi\, \Gamma(-a)\, (x^+x^-)^{1+a}} \tag{B.2}$$

[9] Note that the Fourier transform integral of the y-integration cancels out by the customary momentum conservation δ-function.

This formula may be derived by changing variables to polar coordinates, integrating out the angular variable to produce a Bessel function J_0, and carrying out the radial integration in terms of tabulated integrals. Successive differentiation in x^+ gives,

$$\int \frac{d^2p}{(2\pi)^2} e^{ip\cdot x} (p^2)^a p_+^{2n} = \frac{(-)^n \, 2^{2a} \, \Gamma(a+1+2n)}{\pi \, \Gamma(-a) \, (x^+)^{2n} \, (x^+x^-)^{a+1}} \tag{B.3}$$

Evaluating this expression at $a=-1$, and at $a=0$ gives,

$$\begin{aligned} \int \frac{d^2p}{(2\pi)^2} e^{ipx} \frac{p_+^{2n}}{p^2} &= \frac{(-)^n \Gamma(2n)}{4\pi (x^+)^{2n}} \\ \int \frac{d^2p}{(2\pi)^2} e^{ipx} p_+^{2n} \ln(p^2) &= \frac{(-)^{n+1}\Gamma(2n+1)}{\pi (x^+)^{2n} (x^+x^-)} \end{aligned} \tag{B.4}$$

8. Appendix C: Positivity of the current algebra level in unitary theories

Take the simple example of a charged scalar field ϕ in the presence of a U(1)-gauge field A_μ in Euclidean 2-dimensional space, with Euclidean action,

$$S = \int d^2x \, |\partial_\mu \phi - i A_\mu \phi|^2 \tag{8.1}$$

Following our standard definition, $\delta S = -\int d^2x \, J^\mu \delta A_\mu$ the current is found to be,

$$J_\mu = -i \left(\phi^* \partial_\mu \phi - \phi \partial_\mu \phi^* \right) \tag{8.2}$$

Decomposing $\phi = \phi_0 e^{i\theta}$, the free ϕ action reduces to

$$S = \int d^2x \, \left(4\partial_+ \phi_0 \partial_- \phi_0 + 4\phi_0^2 \partial_+ \theta \partial_- \theta \right) \tag{8.3}$$

while the current reduces to $J_\mu = 2\phi_0^2 \partial_\mu \theta$. We now set the ϕ_0 field to a constant, and evaluate the θ two-point function,

$$\langle \theta(x) \theta(y) \rangle = -\frac{1}{8\pi \phi_0^2} \ln |x-y|^2 \tag{8.4}$$

The current two-point function is then readily evaluated, and we find,

$$\langle J_+(x) J_+(y) \rangle = -4\phi_0^4 \partial_+^x \partial_+^y \ln |x-y|^2 = -\frac{\phi_0^2}{2\pi (x^+ - y^+)^2} \tag{8.5}$$

This formula provides the canonical normalization of the level for a free boson.

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